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Directed Krasnosel'skiĭ-Mann Iteration

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Abstract In this article, we study a directed Krasnosel'skiĭ-Mann iteration in real Hilbert spaces. Through the utilization of new, self-contained, and simplified techniques, we prove its weak convergence. Notably, our upper bounds on directed factors are desirably larger than existing ones, assuming that the involved factors remain constant.

Keywords Non-expansive operator, Krasnosel'skiĭ-Mann iteration, Directed factors, Weak convergence
2020 MSC classification: 47H05, 47J20, 47J25.

1 Introduction

It is well-known that fixed point theory and algorithms play a crucial role in many branches of mathematics.

The Krasnosel'skiĭ-Mann (KM for short) iteration [12, 13, 17, 14] serves as a fundamental iterative scheme for locating fixed points of non-expansive operators in real Hilbert spaces. The distinctive characteristic of the KM iteration is that the new iterate is obtained as a convex combination of the current iterate and its operator evaluation.

Recently, [5, 8] investigated a directed version of the KM iteration, which can be viewed as a special case of [15], incorporating an additional term that involves a nonnegative factor and the difference between the two most recent iterates. Notably, the assumptions [8] regarding this factor are less restrictive than those assumptions [5], and they are independent of the iterates themselves.

The objective of this article is to introduce significantly different assumptions regarding the aforementioned factor. As illustrated below, these assumptions are weaker than those [8] and remarkably enable a new, concise, self-contained, and simplified proof of the weak convergence of the directed KM iteration. Impressively, this proof no longer relies on the seminal lemma [1].

2 Preliminaries

In this section, we first give some basic definitions and then provide some auxiliary results for later use.

Let $\mathcal{H}$ be an infinite-dimensional real Hilbert space equipped with the standard inner product $\langle x, y\rangle$ and the induced norm $|x| = \sqrt{\langle x, x\rangle}$ for $x, y \in \mathcal{H}$.

An operator $T : \mathcal{H} \to \mathcal{H}$ is said to be non-expansive if it satisfies the inequality:

$$\|T(x) - T(y)\| \leq \|x - y\|$$

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for all x, y in $\mathcal{H}$.

The Krasnosel'skiĭ-Mann (KM) iteration is defined as follows:

$$x_{k+1} = (1-\alpha_k)x_k + \alpha_k T(x_k), \quad k = 0, 1, \dots,$$

where the coefficient $\alpha_k \in [0,1]$ and the series $\sum \alpha_k(1-\alpha_k)$ diverges. Refer to [14, 11] for relevant discussions and the cited references.

To speed up the KM iteration, [5] proposed the following modification:

$$x_{k+1} = (1-\alpha_k)x_k + \alpha_k T(x_k) + (1-\alpha_k)\delta_k(x_k - x_{k-1}),$$

where $\delta_k \geq 0$ and $x_{-1} := x_0$. Hereafter, $(1-\alpha_k)\delta_k$ is called a *directed factor*.

For weak convergence, the following assumptions were introduced in [8]:

$$\varepsilon \leq \alpha_k < 1, \tag{1}$$

$$0 \leq \delta_k \leq 1, \tag{2}$$

$$(1-\alpha_{k-1})\delta_{k-1} \leq (1-\alpha_k)\delta_k, \tag{3}$$

$$\left(\frac{1}{\alpha_{k-1}} - 1\right)(1-\delta_{k-1}) - \left(2 - \frac{1}{\alpha_k} - \alpha_k\right)\delta_k^2 - \left(\frac{1}{\alpha_k} - \alpha_k\right)\delta_k \geq \varepsilon, \tag{4}$$

where ε is a given sufficiently small positive number.

Finally, we give the following two lemmas to be used later.

Lemma 2.1 *Assume that $\alpha > 0$. If $4\alpha\beta \geq \gamma^2$, then*

$$\alpha\|a\|^2 + \beta\|b\|^2 + \gamma\langle a, b\rangle \geq 0, \quad \forall a, b \in \mathcal{H}.$$

Lemma 2.2 *[3, 16] Consider any maximal monotone operator $A\colon \mathcal{H} \rightrightarrows \mathcal{H}$. Assume that the sequence $\{w^k\}$ in $\mathcal{H}$ converges weakly to w, and the sequence $\{s^k\}$ on domT converges strongly to s. If $A(w^k) \ni s^k$ for any k, then the relation $A(w) \ni s$ must hold.*

3 Directed KM Iteration

In this section, we propose directed KM iteration, and we suggest new and weaker assumptions for analyzing weak convergence.

First of all, we make the following assumptions. For a given sufficiently small positive number ε, we assume that

$$\varepsilon \leq \alpha_k \leq 1-\varepsilon, \tag{5}$$

$$0 \leq \delta_k \leq \frac{1}{1-\alpha_k}\left(1 - \varepsilon - \frac{1}{1+(1-\sigma)(1-\alpha_{k-1})/\alpha_{k-1}}\right), \tag{6}$$

$$(1-\alpha_{k-1})\delta_{k-1} \leq (1-\alpha_k)\delta_k, \tag{7}$$

$$\delta_k \leq \frac{-1 + \sqrt{1 + 4((\frac{1}{\sigma}-1)\frac{1}{\alpha_k}+1)((1-\sigma)\frac{1-\alpha_{k-1}}{\alpha_{k-1}}\frac{1}{1-\alpha_k} - \frac{\varepsilon}{1-\alpha_k})}}{2(\frac{1}{\sigma}-1)\frac{1}{\alpha_k}+2}, \tag{8}$$

where σ is chosen in the interval $[0.01, 0.99]$.

Below we describe the aforementioned directed KM iteration — Algorithm 3.1.

Algorithm 3.1 directed KM iteration

1: Choose $x_{-1} = x_0 \in \mathcal{H}$. Choose some sufficiently small positive number ε. Set $k := 0$.

2: Choose (α_k, δ_k) such that (5)-(8) hold. Compute

$$x_{k+1} = (1-\alpha_k)x_k + \alpha_k T(x_k) + (1-\alpha_k)\delta_k(x_k - x_{k-1}). \tag{9}$$

Set $k := k+1$.

Next, we numerically demonstrate the assumptions (6) and (8) to some extent. For brevity, we simply set $\alpha_k \equiv \alpha,\ \delta_k \equiv \delta$. Thus, we get

$$\delta < \min\left\{\frac{1}{1-\alpha}\left(1-\frac{1}{1+(1-\sigma)(1-\alpha)/\alpha}\right),\ \frac{-1+\sqrt{1+4((\frac{1}{\sigma}-1)\frac{1}{\alpha}+1)(1-\sigma)\frac{1}{\alpha}}}{2(\frac{1}{\sigma}-1)\frac{1}{\alpha}+2}\right\}, \tag{10}$$

where σ is chosen in $[0.01, 0.99]$.

Numerical demonstration was given in Table 1, where δ_- corresponds to the values from [8, Table 1], the pair $(\delta(10), \sigma)$ corresponds to (10).

Table 1: Numerical comparisons of [8, Table 1] with (10)

α	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	0.99
δ_-	0.4105	0.3983	0.3870	0.3765	0.3668	0.3576	0.3490	0.3410	0.3348
$\delta(10)$	0.4397	0.4230	0.4075	0.3930	0.3795	0.3668	0.3549	0.3437	0.3353
σ	0.49	0.46	0.45	0.42	0.40	0.38	0.36	0.34	0.33
α	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55
δ_-	0.6143	0.5746	0.5426	0.5157	0.4927	0.4725	0.4545	0.4384	0.4239
$\delta(10)$	0.6394	0.6389	0.6038	0.5730	0.5455	0.5206	0.4978	0.4769	0.4575
σ	0.75	0.70	0.66	0.63	0.61	0.58	0.56	0.54	0.50

From Table 1, we can observe that our computed values of δ are consistently larger than the corresponding values from [8, Table 1] for each sampling point.

Remark 3.1 The manuscript entitled *On an accelerated Krasnosel'skiĭ-Mann iteration*, mentioned in subsequent work [7], is an original draft of this article. In the current version, we have revised "accelerated" as "directed" because the term $x_k - x_{k-1}$ in Algorithm 3.1 can provide information on previous descent/search *direction*. In this sense, the terminology "directed", corresponding to "direction", appears to be more appropriate.

4 Weak convergence

In this section, we prove weak convergence of Algorithm 3.1.

Lemma 4.1 *Let $\{x_k\}$ be the sequence generated by Algorithm 3.1, and let z be a fixed point of T. Then*

$$\begin{aligned}\|x_{k+1}-z\|^2 \le{}& (1+(1-\alpha_k)\delta_k)\|x_k-z\|^2-(1-\alpha_k)\delta_k\|x_{k-1}-z\|^2\\ &-\frac{1-\alpha_k}{\alpha_k}\|x_{k+1}-x_k\|^2+2\frac{1-\alpha_k}{\alpha_k}\delta_k\langle x_{k+1}-x_k, x_k-x_{k-1}\rangle\\ &+\left((1-\alpha_k)\delta_k(1+\delta_k)-\frac{1-\alpha_k}{\alpha_k}\delta_k^2\right)\|x_k-x_{k-1}\|^2.\end{aligned}$$

Proof In view of the iterative formula (9), we have

$$\hat{x}_k = x_k + \delta_k(x_k - x_{k-1}), \tag{11}$$

$$x_{k+1} = (1-\alpha_k)\hat{x}_k + \alpha_k T(x_k). \tag{12}$$

For any given fixed point z of T, i.e., $T(z) = z$, it follows from (12) that

$$x_{k+1} - z = (1-\alpha_k)(\hat{x}_k - z) + \alpha_k(Tx_k - Tz).$$

Since T is non-expansive, we have

$$\begin{aligned}\|x_{k+1}-z\|^2 &= \alpha_k\|Tx_k-Tz\|^2+(1-\alpha_k)\|\hat{x}_k-z\|^2-\alpha_k(1-\alpha_k)\|Tx_k-\hat{x}_k\|^2\\ &\le \alpha_k\|x_k-z\|^2+(1-\alpha_k)\|\hat{x}_k-z\|^2-\alpha_k(1-\alpha_k)\|Tx_k-\hat{x}_k\|^2.\end{aligned}$$

From (12) and (11), we have

$$\alpha_k (Tx_k - \hat{x}_k) = x_{k+1} - \hat{x}_k = x_{k+1} - x_k - \delta_k(x_k - x_{k-1}),$$

so, we get

$$\begin{aligned}&\alpha_k^2 \|Tx_k - \hat{x}_k\|^2 \\ &= \|x_{k+1} - x_k\|^2 + \delta_k^2 \|x_k - x_{k-1}\|^2 - 2\delta_k \langle x_{k+1} - x_k, x_k - x_{k-1}\rangle.\end{aligned}$$

Observe that

$$\begin{aligned}\|\hat{x}_k - z\|^2 &= \|(1+\delta_k)(x_k - z) - \delta_k(x_{k-1} - z)\|^2 \\ &= (1+\delta_k)\|x_k - z\|^2 - \delta_k\|x_{k-1} - z\|^2 + \delta_k(1+\delta_k)\|x_k - x_{k-1}\|^2.\end{aligned}$$

Thus, we further get

$$\begin{aligned}\|x_{k+1} - z\|^2 \le\ & (1 + (1-\alpha_k)\delta_k)\|x_k - z\|^2 - (1-\alpha_k)\delta_k\|x_{k-1} - z\|^2 \\ & - \frac{1-\alpha_k}{\alpha_k}\|x_{k+1} - x_k\|^2 + 2\frac{1-\alpha_k}{\alpha_k}\delta_k\langle x_{k+1} - x_k, x_k - x_{k-1}\rangle \\ & + \left((1-\alpha_k)\delta_k(1+\delta_k) - \frac{1-\alpha_k}{\alpha_k}\delta_k^2\right)\|x_k - x_{k-1}\|^2.\end{aligned}$$

The proof is complete. □

Theorem 4.1 *If α_k and δ_k further satisfy (5)-(8), then the sequence $\{x_k\}$ generated by Algorithm 3.1 is weakly convergent.*

Proof It follows from Lemma 4.1 and (7) that

$$\begin{aligned}&\|x_{k+1} - z\|^2 - (1-\alpha_{k+1})\delta_{k+1}\|x_k - z\|^2 + (1-\sigma)\frac{1-\alpha_k}{\alpha_k}\|x_{k+1} - x_k\|^2 \\ &\le \|x_k - z\|^2 - (1-\alpha_k)\delta_k\|x_{k-1} - z\|^2 + (1-\sigma)\frac{1-\alpha_{k-1}}{\alpha_{k-1}}\|x_k - x_{k-1}\|^2 - \Delta_k, \end{aligned} \tag{13}$$

where $\sigma \in (0,1)$ and Δ_k is given by

$$\begin{aligned}\Delta_k :=\ & \sigma\frac{1-\alpha_k}{\alpha_k}\|x_{k+1} - x_k\|^2 - 2\frac{1-\alpha_k}{\alpha_k}\delta_k\langle x_{k+1} - x_k, x_k - x_{k-1}\rangle \\ & + \left((1-\sigma)\frac{1-\alpha_{k-1}}{\alpha_{k-1}} - (1-\alpha_k)\delta_k(1+\delta_k) + \frac{1-\alpha_k}{\alpha_k}\delta_k^2\right)\|x_k - x_{k-1}\|^2.\end{aligned}$$

Set

$$\varphi_k := \|x_k - z\|^2 - (1-\alpha_k)\delta_k\|x_{k-1} - z\|^2 + (1-\sigma)\frac{1-\alpha_{k-1}}{\alpha_{k-1}}\|x_k - x_{k-1}\|^2. \tag{14}$$

Then

$$\varphi_{k+1} \le \varphi_k - \Delta_k. \tag{15}$$

Consider

$$\begin{aligned}\varphi_k :&= \|x_k - z\|^2 - (1-\alpha_k)\delta_k\|x_{k-1} - z\|^2 + (1-\sigma)\frac{1-\alpha_{k-1}}{\alpha_{k-1}}\|x_k - x_{k-1}\|^2 \\ &= \|x_{k-1} - z\|^2 + 2\langle x_{k-1} - z, x_k - x_{k-1}\rangle + \|x_k - x_{k-1}\|^2 \\ &\quad - (1-\alpha_k)\delta_k\|x_{k-1} - z\|^2 + (1-\sigma)\frac{1-\alpha_{k-1}}{\alpha_{k-1}}\|x_k - x_{k-1}\|^2 \\ &= (1 - (1-\alpha_k)\delta_k)\|x_{k-1} - z\|^2 + 2\langle x_{k-1} - z, x_k - x_{k-1}\rangle \\ &\quad + \left(1 + (1-\sigma)\frac{1-\alpha_{k-1}}{\alpha_{k-1}}\right)\|x_k - x_{k-1}\|^2.\end{aligned}$$

Combining this with Lemma 4.1 and the assumption (6)

$$\begin{aligned}&\delta_k \le \frac{1}{1-\alpha_k}\left(1 - \varepsilon - \frac{1}{1 + (1-\sigma)(1-\alpha_{k-1})/\alpha_{k-1}}\right) \\ \Leftrightarrow\quad & (1 - \varepsilon - (1-\alpha_k)\delta_k)(1 + (1-\sigma)(1-\alpha_{k-1})/\alpha_{k-1}) \ge 1\end{aligned}$$

yields

$$\varphi_k \geq \varepsilon \|x_{k-1} - z\|^2.$$

Similarly, by Lemma 4.1 and the assumption (8)

$$\begin{aligned}
\delta_k \leq & \frac{-1 + \sqrt{1 + 4((\frac{1}{\sigma} - 1)\frac{1}{\alpha_k} + 1)((1-\sigma)\frac{1-\alpha_{k-1}}{\alpha_{k-1}}\frac{1}{1-\alpha_k} - \frac{\varepsilon}{1-\alpha_k})}}{2(\frac{1}{\sigma} - 1)\frac{1}{\alpha_k} + 2} \\
\Leftrightarrow \quad & \left((\frac{1}{\sigma} - 1)\frac{1}{\alpha_k} + 1\right)\delta_k^2 + \delta_k - (1-\sigma)\frac{1-\alpha_{k-1}}{\alpha_{k-1}}\frac{1}{1-\alpha_k} + \frac{\varepsilon}{1-\alpha_k} \leq 0 \\
\Leftrightarrow \quad & \sigma\frac{1-\alpha_k}{\alpha_k}\left((1-\sigma)\frac{1-\alpha_{k-1}}{\alpha_{k-1}} - (1-\alpha_k)\delta_k(1+\delta_k) + \frac{1-\alpha_k}{\alpha_k}\delta_k^2 - \varepsilon\right) \\
\geq & \frac{(1-\alpha_k)^2}{\alpha_k^2}\delta_k^2,
\end{aligned}$$

we can get

$$\varDelta_k \geq \varepsilon \|x_k - x_{k-1}\|^2.$$

Obviously, from these two relations and (15), we conclude that

$$\lim\ \varphi_k \text{ exists} \quad \Rightarrow \quad \|x_{k-1} - z\| \text{ (thus } \|x_k - z\|\text{) is bounded;} \tag{16}$$

$$\lim\ \varDelta_k = 0 \quad \Rightarrow \quad \lim \|x_k - x_{k-1}\| = 0. \tag{17}$$

From $\alpha_k \in [\varepsilon, 1-\varepsilon]$ and

$$(I - T)(x_k) = \frac{(1-\alpha_k)\delta_k(x_k - x_{k-1}) - (x_{k+1} - x_k)}{\alpha_k},$$

On the other hand, as proved in (16), $\{x_k\}$ is bounded in norm, thus there exists at least one weak cluster point z^∞, i.e.,

$$x_{k_j} \rightharpoonup z^\infty. \tag{18}$$

Finally, since $I - N$ is continuous and monotone, it must be maximal monotone. In view of Lemma 2.2, $(I - N)(z^\infty) = 0$.

Denote

$$\phi_k(z) := \|x_k - z\|^2 - (1-\alpha_k)\delta_k\|x_{k-1} - z\|^2.$$

Then, it follows from (14), (16) and (17) that $\lim \phi_k(z)$ exists as well.

Below, we show that $\{x_k\}$ weakly converges to z^∞. Let z_1^∞ and z_2^∞ be two weak cluster points of $\{x_k\}$. Then, repeating the arguments above yields that z_1^∞ and z_2^∞ are solutions. Correspondingly, we set

$$l_i := \lim \phi_k(z_i^\infty), \quad i = 1, 2.$$

Consider

$$\begin{aligned}
&\|x_k - z_1^\infty\|^2 - (1-\alpha_k)\delta_k\|x_{k-1} - z_1^\infty\|^2 \\
&= \|x_k - z_2^\infty\|^2 - 2\langle x_k - z_2^\infty, z_1^\infty - z_2^\infty\rangle + \|z_1^\infty - z_2^\infty\|^2 \\
&\quad - (1-\alpha_k)\delta_k\left(\|x_{k-1} - z_2^\infty\|^2 - 2\langle x_{k-1} - z_2^\infty, z_1^\infty - z_2^\infty\rangle + \|z_1^\infty - z_2^\infty\|^2\right) \\
&= \|x_k - z_2^\infty\|^2 - (1-\alpha_k)\delta_k\|x_{k-1} - z_2^\infty\|^2 - 2\langle x_k - z_2^\infty, z_1^\infty - z_2^\infty\rangle + 2(1-\alpha_k)\delta_k\langle x_{k-1} - z_2^\infty, z_1^\infty - z_2^\infty\rangle \\
&\quad + (1 - (1-\alpha_k)\delta_k)\|z_1^\infty - z_2^\infty\|^2 \\
&\geq \|x_k - z_2^\infty\|^2 - (1-\alpha_k)\delta_k\|x_{k-1} - z_2^\infty\|^2 - 2\langle x_k - z_2^\infty, z_1^\infty - z_2^\infty\rangle + 2(1-\alpha_k)\delta_k\langle x_{k-1} - z_2^\infty, z_1^\infty - z_2^\infty\rangle \\
&\quad + \varepsilon\|z_1^\infty - z_2^\infty\|^2,
\end{aligned}$$

where the inequality follows from (6), which indicates

$$(1-\alpha_k)\delta_k \leq 1 - \varepsilon.$$

Meanwhile

$$\langle x_{k-1} - z_2^\infty, z_1^\infty - z_2^\infty\rangle = \langle x_k - z_2^\infty, z_1^\infty - z_2^\infty\rangle - \langle x_k - x_{k-1}, z_1^\infty - z_2^\infty\rangle.$$

Combing this with (17) and taking the limit along $k \in \mathcal{N}_1$, where $\mathcal{N}_1$ such that $\{x_k\}$ along $k \in \mathcal{N}_1$ weakly converges to z_2^∞, we get

$$l_1 \geq l_2 + \varepsilon \|z_1^\infty - z_2^\infty\|^2.$$

Similarly, we also get

$$l_2 \geq l_1 + \varepsilon \|z_2^\infty - z_1^\infty\|^2.$$

Adding these two inequalities yields $z_1^\infty = z_2^\infty$ and $\{x_k\}$ converges weakly. □

5 Conclusions

In this article, we have studied a directed Krasnosel'skiĭ-Mann iteration in real Hilbert spaces. By invoking new, self-contained, and simplified techniques, we prove its weak convergence. Notably, our new upper bounds on directed factors are desirably larger than existing ones, assuming that the involved factors remain constant.

At the end of this article, it shall be specially stressed that our new techniques would be a widely used tool in enlarging upper bounds on directed factors in context of some splitting methods (cf. [8, 4]). The resulting directed splitting methods can be more efficient in practice [8, 4] than those inertial ones (cf. [2, 6, 11, 10]), but with more direct and much simpler convergence analysis.

Availability of Data and Material Not applicable.

Declarations

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References

1. Alvarez, F.: Weak convergence of a relaxed and inertial hybrid projection-proximal point algorithm for maximal monotone operators in Hilbert space. SIAM J. Optim. **14**(3), 773-782 (2004)
2. Alves, M.M., Eckstein, J., Geremia, M., Melo, J.G.: Relative-error inertial-relaxed inexact versions of Douglas-Rachford and ADMM splitting algorithms. Comput. Optim. Appl. **75**, 389-422 (2020)
3. Brézis, H.: Opérateurs maximaux monotones. North-Holland, Amsterdam (1973)
4. Chen, M.M.: Directed Douglas-Rachford Splitting. Thesis, Zhengzhou University (2024)
5. Dong, Q.L., Yuan, H.B.: Accelerated Mann and CQ algorithms for finding a fixed point of a nonexpansive mapping. Fixed Point Theory Appl. **2015**, 125 (2015)
6. Dong, Y.D.: New inertial factors of the Krasnosel'skiĭ-Mann iteration. Set-Valued Var. Anal. **29**(1), 145-161 (2021)
7. Dong, Y.D., Luo, Q.Q.: On an inertial Krasnosel'skiĭ-Mann iteration. Appl. Set-Valued Anal. Optim. **6**(1), 103-112 (2024)
8. Dong, Y.D., Sun, M.D.: New acceleration factors of the Krasnosel'skiĭ-Mann iteration. Results Math. **77**, 194 (2022)
9. Eckstein, J., Svaiter, B.F.: A family of projective splitting methods for the sum of two maximal monotone operators. Math. Program. **111**, 173-199 (2008)
10. Fierro, I., Maulen, J.J., Peypouquet, J.: Inertial Krasnosel'skiĭ-Mann iterations. arXiv preprint arXiv:2210.03791v3 (2023)
11. Izuchukwu, C., Shehu, Y.: Inertial reflected Krasnosel'skiĭ-Mann iteration with applications to monotone inclusion problems. Optimization (Accepted)
12. Krasnosel'skiĭ, M.A.: Two remarks on the method of successive approximations. Uspekhi. Mat. Nauk. **10**, 123-127 (1955)
13. Mann, W.R.: Mean value methods in iteration. Am. Math. Soc. **4**, 506-510 (1953)
14. Matsushita, S.Y.: On the convergence rate of the Krasnosel'skiĭ-Mann iteration. Bull. Aust. Math. Soc. **96**(1), 162-170 (2017)
15. Moudafi, A., Oliny, M.: Convergence of a splitting inertial proximal method for monotone operators. J. Comput. Appl. Math. **155**(2), 447-454 (2003)
16. Pascali, D., Sburlan, S.: Nonlinear mappings of monotone type. Editura Academiei, Bucharest (1978)
17. Reich, S.: Weak convergence theorems for nonexpansive mappings in Banach spaces. J. Math. Anal. Appl. **67**, 274-276 (1979)

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On an accelerated Krasnoselskii-Mann iteration

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1 Introduction

In many branches of mathematics, fixed point theory and algorithms play a crucial role as fundamental tools.

The Krasnosel'skiĭ-Mann (KM) iteration [1, 2] serves as a fundamental iterative scheme for locating fixed points of non-expansive operators in real Hilbert spaces. The distinctive characteristic of the KM iteration is that the new iterate is obtained as a convex combination of the current iterate and its operator evaluation.

Recently, [3, 4] investigated an accelerated version of the KM iteration, incorporating an additional term that involves a nonnegative factor and the difference between the two most recent iterates. Notably, the assumptions [4]

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regarding this factor are less restrictive compared to those [3], and they are independent of the iterates themselves.

The objective of this note is to introduce significantly different assumptions regarding the aforementioned factor. As illustrated below, these assumptions are weaker than those [4] and remarkably enable a new, concise, self-contained, and simplified proof of the weak convergence of the accelerated KM iteration. Impressively, unlike the proof of [4, Theorem 1], this proof no longer relies on the seminal result [5, Lemma 2.3].

2 Preliminaries

In this section, we first give some basic definitions and then provide some auxiliary results for later use.

Let $\mathcal{H}$ be an infinite-dimensional real Hilbert space equipped with the standard inner product $\langle x, y\rangle$ and the induced norm $|x| = \sqrt{\langle x, x\rangle}$ for $x, y \in \mathcal{H}$.

An operator $T : \mathcal{H} \to \mathcal{H}$ is said to be non-expansive if it satisfies the inequality:

$$\|T(x) - T(y)\| \leq \|x - y\|$$

for all x, y in $\mathcal{H}$.

The Krasnosel'skiĭ-Mann (KM) iteration is defined as follows:

$$x_{k+1} = (1 - \alpha_k)x_k + \alpha_k T(x_k), \quad k = 0, 1, \ldots,$$

where the coefficient $\alpha_k \in [0, 1]$ and the series $\sum \alpha_k(1 - \alpha_k)$ diverges. Refer to [6–8] for relevant discussions and the cited references.

To accelerate the KM iteration, [3] proposed the following modification:

$$x_{k+1} = (1 - \alpha_k)x_k + \alpha_k T(x_k) + (1 - \alpha_k)\delta_k(x_k - x_{k-1}),$$

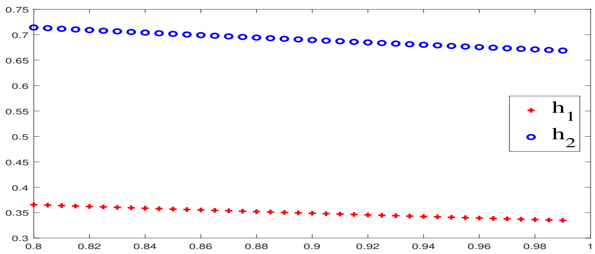
where $\delta_k \geq 0$ and $x_{-1} := x_0$.

For weak convergence, the following assumptions were introduced in [4]:

$$\begin{aligned} &\varepsilon \leq \alpha_k < 1, \quad 0 \leq \delta_k \leq 1, \quad (1 - \alpha_{k-1})\delta_{k-1} \leq (1 - \alpha_k)\delta_k, \\ &\left(\frac{1}{\alpha_{k-1}} - 1\right)(1 - \delta_{k-1}) - \left(2 - \frac{1}{\alpha_k} - \alpha_k\right)\delta_k^2 - \left(\frac{1}{\alpha_k} - \alpha_k\right)\delta_k \geq \varepsilon, \end{aligned} \tag{1}$$

where ε is a given sufficiently small positive number.

Fig. 1: The graphs of h_1 and h_2



3 Results

In this section, we propose accelerated KM iteration, and we suggest new and weaker assumptions for analyzing weak convergence.

First of all, we would like to point out that, by our numerical experiments [4], α_k in the KM iteration shall be close to 1 for numerical efficiency in practice and weak convergence in theory. Thus, we give a practical, accelerated KM iteration — Algorithm 3.1.

Algorithm 3.1 a practical, accelerated KM iteration

1: Choose $x_{-1} = x_0 \in \mathcal{H}$. Choose $\varepsilon = 10^{-9}$ and $\alpha \in [0.80, 0.99]$. Compute $h_1(\alpha)$ via (2), and denote by δ^+. Choose $\delta_{-1} = 0$. Set $k := 0$.

2: Choose $\delta_k \in [\delta_{k-1}, \delta^+]$. Compute

$$x_{k+1} = (1-\alpha)x_k + \alpha T(x_k) + (1-\alpha)\delta_k(x_k - x_{k-1}).$$

Set $k := k+1$.

To provide a better understanding of the practical, accelerated KM iteration, we define $h_1(\alpha)$ and $h_2(\alpha)$ as follows.

$$h_1(\alpha) := 0.5\,\frac{-1+\sqrt{1+4(\frac{2}{\alpha}+1)(\frac{2}{3\alpha}-\frac{\varepsilon}{1-\alpha})}}{\frac{2}{\alpha}+1}, \tag{2}$$

$$h_2(\alpha) := \frac{1}{1-\alpha}\left(1-\varepsilon-\frac{1}{1+\frac{2}{3}\frac{1-\alpha}{\alpha}}\right),$$

where $\varepsilon = 10^{-9}$, and the graphs of $h_1(\alpha)$ and $h_2(\alpha)$ are plotted in Fig. 1 using MATLAB. Also, it is direct to check that

$$\frac{1}{3} < h_1(\alpha) < \frac{1}{2} < h_2(\alpha), \quad \alpha \in [0.80, 0.99]. \tag{3}$$

Below, we describe the accelerated KM iteration in a general case of $\alpha_k \in [\varepsilon, 1-\varepsilon]$, for a given sufficiently small positive number ε.

For the following accelerated KM iteration

$$x_{k+1} = (1-\alpha_k)x_k + \alpha_k T(x_k) + (1-\alpha_k)\delta_k(x_k - x_{k-1}), \quad k = 0, 1, ..., \tag{4}$$

with $\alpha_{-1} = \alpha_0$ in $[\varepsilon, 1-\varepsilon]$ and $\delta_{-1} = 0$, we assume that, (i) the sequences $\{\alpha_k\}$ and $\{\delta_k\}$ satisfy

$$\alpha_k \in [\varepsilon, 1-\varepsilon], \quad \delta_k \geq \delta_{k-1}(1-\alpha_{k-1})/(1-\alpha_k); \tag{5}$$

(ii)

$$\delta_k^+ := 0.5\,\frac{-1+\sqrt{1+4((\frac{1}{\sigma}-1)\frac{1}{\alpha_k}+1)((1-\sigma)\frac{1-\alpha_{k-1}}{\alpha_{k-1}}\frac{1}{1-\alpha_k}-\frac{\varepsilon}{1-\alpha_k})}}{(\frac{1}{\sigma}-1)\frac{1}{\alpha_k}+1},$$

$$\delta_k \leq \min\left\{\delta_k^+,\, \frac{1}{1-\alpha_k}\left(1-\varepsilon-\frac{1}{1+(1-\sigma)(1-\alpha_{k-1})/\alpha_{k-1}}\right)\right\}, \tag{6}$$

where σ is chosen in $(0,1)$ in advance.

Unlike (1), the corresponding assumption (6) no longer includes δ_{k-1}. This fully shows that they are widely different.

Obviously, for this accelerated KM iteration described by (4)-(6), it reduces to Algorithm 3.1 provided that $\alpha_k \equiv \alpha$ and $\sigma = 1/3$.

Notice that, the extra variable σ in (6) shall be chosen to maximize the min function defined in the interval $(0,1)$. See Remark 3.2 and Fig. 2 below for more details.

In the analysis of weak convergence for the accelerated KM iteration given by (4)-(6), we make use of the following lemmas to simplify the analysis.

Lemma 3.1 *Assume that $\alpha > 0$. If $4\alpha\beta \geq \gamma^2$, then*

$$\alpha\|a\|^2 + \beta\|b\|^2 + \gamma\langle a, b\rangle \geq 0, \quad \forall a, b \in \mathcal{H}.$$

Lemma 3.2 *([9, Sect. 3]) Let $\alpha > 0$, $t \in \mathcal{R}$. If $4\alpha > t^2\beta$, then the following*

$$\langle x, \alpha x\rangle + \langle u, \beta u\rangle - t\langle x, \beta u\rangle \geq \frac{\alpha+\beta-\sqrt{(\alpha-\beta)^2+t^2\beta^2}}{2}\left(\|x\|^2+\|u\|^2\right)$$

holds for all x, $u \in \mathcal{H}$.

Using these lemmas, we can establish the weak convergence of the accelerated KM iteration.

Theorem 3.1 *The sequence $\{x_k\}$ generated by Algorithm 3.1 converges weakly to a fixed point of T.*

For Theorem 3.1, we decide to omit its proof details here. This is because that (i) it can be viewed as an instance $\sigma = 1/3$ of that of the next theorem and (ii) the statement "Choose $\delta_k \in [\delta_{k-1}, \delta^+]$" in Algorithm 3.1 corresponds to (5) and (6); see Fig. 1 and the desired inequality (3).

Theorem 3.2 *If the assumptions (5)-(6) hold, then the sequence $\{x_k\}$ generated by (4) converges weakly to a fixed point of T.*

Proof In view of (4), we have

$$\hat{x}_k = x_k + \delta_k (x_k - x_{k-1}), \tag{7}$$

$$x_{k+1} = (1-\alpha_k)\hat{x}_k + \alpha_k T(x_k). \tag{8}$$

For any given fixed point z of T, i.e., $T(z) = z$, it follows from (8) that

$$x_{k+1} - z = (1-\alpha_k)(\hat{x}_k - z) + \alpha_k (Tx_k - Tz).$$

Since T is non-expansive, we have

$$\begin{aligned}\|x_{k+1} - z\|^2 &= \alpha_k \|Tx_k - Tz\|^2 + (1-\alpha_k)\|\hat{x}_k - z\|^2 - \alpha_k(1-\alpha_k)\|Tx_k - \hat{x}_k\|^2 \\ &\le \alpha_k \|x_k - z\|^2 + (1-\alpha_k)\|\hat{x}_k - z\|^2 - \alpha_k(1-\alpha_k)\|Tx_k - \hat{x}_k\|^2.\end{aligned}$$

From (8) and (7), we have

$$\alpha_k (Tx_k - \hat{x}_k) = x_{k+1} - \hat{x}_k = x_{k+1} - x_k - \delta_k(x_k - x_{k-1}),$$

so, we get

$$\begin{aligned}&\alpha_k^2 \|Tx_k - \hat{x}_k\|^2 \\ &= \|x_{k+1} - x_k\|^2 + \delta_k^2 \|x_k - x_{k-1}\|^2 - 2\delta_k \langle x_{k+1} - x_k, x_k - x_{k-1}\rangle.\end{aligned}$$

Observe that

$$\begin{aligned}\|\hat{x}_k - z\|^2 &= \|(1+\delta_k)(x_k - z) - \delta_k(x_{k-1} - z)\|^2 \\ &= (1+\delta_k)\|x_k - z\|^2 - \delta_k \|x_{k-1} - z\|^2 + \delta_k(1+\delta_k)\|x_k - x_{k-1}\|^2.\end{aligned}$$

Thus, we further get

$$\begin{aligned}\|x_{k+1} - z\|^2 \le\ & (1 + (1-\alpha_k)\delta_k)\|x_k - z\|^2 - (1-\alpha_k)\delta_k \|x_{k-1} - z\|^2 \\ & - \frac{1-\alpha_k}{\alpha_k}\|x_{k+1} - x_k\|^2 + 2\frac{1-\alpha_k}{\alpha_k}\delta_k \langle x_{k+1} - x_k, x_k - x_{k-1}\rangle \\ & + \left((1-\alpha_k)\delta_k(1+\delta_k) - \frac{1-\alpha_k}{\alpha_k}\delta_k^2\right)\|x_k - x_{k-1}\|^2.\end{aligned}$$

From the assumption (5), we have

$$\begin{aligned} &\|x_{k+1}-z\|^2-(1-\alpha_{k+1})\delta_{k+1}\|x_k-z\|^2+(1-\sigma)\frac{1-\alpha_k}{\alpha_k}\|x_{k+1}-x_k\|^2 \\ \le\ &\|x_k-z\|^2-(1-\alpha_k)\delta_k\|x_{k-1}-z\|^2+(1-\sigma)\frac{1-\alpha_{k-1}}{\alpha_{k-1}}\|x_k-x_{k-1}\|^2-\Delta_k, \end{aligned} \tag{9}$$

where $\sigma\in(0,1)$ and Δ_k is given by

$$\begin{aligned} \Delta_k :=\ &\sigma\frac{1-\alpha_k}{\alpha_k}\|x_{k+1}-x_k\|^2-2\frac{1-\alpha_k}{\alpha_k}\delta_k\langle x_{k+1}-x_k, x_k-x_{k-1}\rangle \\ &+\left((1-\sigma)\frac{1-\alpha_{k-1}}{\alpha_{k-1}}-(1-\alpha_k)\delta_k(1+\delta_k)+\frac{1-\alpha_k}{\alpha_k}\delta_k^2\right)\|x_k-x_{k-1}\|^2. \end{aligned}$$

Set

$$\varphi_k := \|x_k-z\|^2-(1-\alpha_k)\delta_k\|x_{k-1}-z\|^2+(1-\sigma)\frac{1-\alpha_{k-1}}{\alpha_{k-1}}\|x_k-x_{k-1}\|^2.$$

Then

$$\varphi_{k+1}\le\varphi_k-\Delta_k. \tag{10}$$

Consider

$$\begin{aligned} \varphi_k &:= \|x_k-z\|^2-(1-\alpha_k)\delta_k\|x_{k-1}-z\|^2+(1-\sigma)\frac{1-\alpha_{k-1}}{\alpha_{k-1}}\|x_k-x_{k-1}\|^2 \\ &= \|x_{k-1}-z\|^2+2\langle x_{k-1}-z, x_k-x_{k-1}\rangle+\|x_k-x_{k-1}\|^2 \\ &\quad -(1-\alpha_k)\delta_k\|x_{k-1}-z\|^2+(1-\sigma)\frac{1-\alpha_{k-1}}{\alpha_{k-1}}\|x_k-x_{k-1}\|^2 \\ &= (1-(1-\alpha_k)\delta_k)\|x_{k-1}-z\|^2+2\langle x_{k-1}-z, x_k-x_{k-1}\rangle \\ &\quad +\left(1+(1-\sigma)\frac{1-\alpha_{k-1}}{\alpha_{k-1}}\right)\|x_k-x_{k-1}\|^2. \end{aligned}$$

Combining this with Lemma 3.1 and the assumption (6)

$$\begin{aligned} &\delta_k\le\frac{1}{1-\alpha_k}\left(1-\varepsilon-\frac{1}{1+(1-\sigma)(1-\alpha_{k-1})/\alpha_{k-1}}\right), \\ \Leftrightarrow\quad &(1-\varepsilon-(1-\alpha_k)\delta_k)\left(1+(1-\sigma)(1-\alpha_{k-1})/\alpha_{k-1}\right)\ge 1 \end{aligned}$$

yields

$$\varphi_k\ge\varepsilon\|x_{k-1}-z\|^2.$$

Similarly, by Lemma 3.1 and the assumption (6)

$$\delta_k \leq 0.5 \frac{-1+\sqrt{1+4((\frac{1}{\sigma}-1)\frac{1}{\alpha_k}+1)((1-\sigma)\frac{1-\alpha_{k-1}}{\alpha_{k-1}}\frac{1}{1-\alpha_k}-\frac{\varepsilon}{1-\alpha_k})}}{(\frac{1}{\sigma}-1)\frac{1}{\alpha_k}+1}$$

$$\Leftrightarrow \left((\frac{1}{\sigma}-1)\frac{1}{\alpha_k}+1\right)\delta_k^2+\delta_k-(1-\sigma)\frac{1-\alpha_{k-1}}{\alpha_{k-1}}\frac{1}{1-\alpha_k}+\frac{\varepsilon}{1-\alpha_k}\leq 0$$

$$\Leftrightarrow \sigma\frac{1-\alpha_k}{\alpha_k}\left((1-\sigma)\frac{1-\alpha_{k-1}}{\alpha_{k-1}}-(1-\alpha_k)\delta_k(1+\delta_k)+\frac{1-\alpha_k}{\alpha_k}\delta_k^2-\varepsilon\right)$$

$$\geq \frac{(1-\alpha_k)^2}{\alpha_k^2}\delta_k^2,$$

we can get

$$\Delta_k \geq \varepsilon\,\|x_k - x_{k-1}\|^2.$$

Obviously, from these two relations and (10), we conclude that

$$\lim \varphi_k \text{ exists} \quad \Rightarrow \quad \|x_{k-1}-z\| \text{ (thus } \|x_k - z\|) \text{ is bounded in norm;}$$

$$\lim \Delta_k = 0 \quad \Rightarrow \quad \lim \|x_k - x_{k-1}\| = 0.$$

From (5) and

$$(I-T)(x_k) = \frac{(1-\alpha_k)\delta_k(x_k-x_{k-1})-(x_{k+1}-x_k)}{\alpha_k},$$

it is not difficult to follow [10, Theorem 3.1] to complete the proof. □

Remark 3.1 Next, we numerically demonstrate the assumption (6) to some extent. For brevity, we simply set $\alpha_k \equiv \alpha$, $\delta_k \equiv \delta$. Then the assumption (6) above reduces to

$$\delta^+ := 0.5 \frac{-1+\sqrt{1+4((\frac{1}{\sigma}-1)\frac{1}{\alpha}+1)(1-\sigma)\frac{1}{\alpha}}}{(\frac{1}{\sigma}-1)\frac{1}{\alpha}+1},$$

$$\delta < \min\left\{\delta^+, \frac{1}{1-\alpha}\left(1-\frac{1}{1+(1-\sigma)(1-\alpha)/\alpha}\right)\right\} := f(\sigma). \quad (11)$$

Be aware that, in contrast to (6), we no longer introduce the extra ε above because we turn to resort to Lemma 3.2. In addition, we have replaced $\leq$ there by $<$ here.

Numerical demonstration of (11) is given in Table 1, where δ_{new} stands for a slightly lower approximation of the maximum of f in (11) with respect to σ. We also provide the values from [4, Table 1] for comparison.

From Tables 1 and 2, we can observe that our computed values of δ_{new} are consistently larger than the corresponding values from [4, Table 1] for each sampling point.

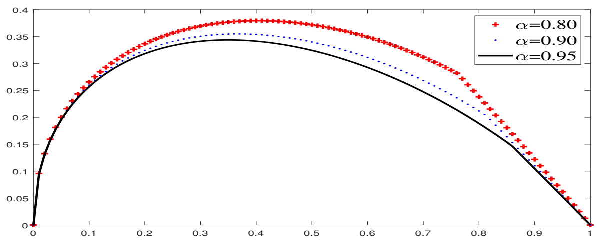
Table 1: Numerical demonstration of (11) with respect to σ

α	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	0.99
δ_{new}	0.4397	0.4230	0.4075	0.3930	0.3795	0.3668	0.3549	0.3437	0.3353
σ	0.49	0.46	0.45	0.42	0.40	0.38	0.36	0.34	0.33
α	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55
δ_{new}	0.6394	0.6389	0.6038	0.5730	0.5455	0.5206	0.4978	0.4769	0.4575
σ	0.75	0.70	0.66	0.63	0.61	0.58	0.56	0.54	0.50

Table 2: Numerical demonstration of [4, Table 1]

α	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	0.99
δ_-	0.4105	0.3983	0.3870	0.3765	0.3668	0.3576	0.3490	0.3410	0.3348
α	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55
δ_-	0.6143	0.5746	0.5426	0.5157	0.4927	0.4725	0.4545	0.4384	0.4239

Fig. 2: The graph of f(σ) for different α



Remark 3.2 For the KM iteration, choosing α close to 1 in its accelerated and inertial versions [4, 10] is generally a good strategy. In this case, it is noted that selecting σ to be equal to or close to 1/3 has been found to be a favorable choice; see Table 1 and Fig. 2.

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Declarations

Conflict of interest Authors declare that they have no conflict of interest.

Data Availability

Has no associated data.

References

1. Krasnosel'skiĭ, M.A.: Two remarks on the method of successive approximations. Uspekhi. Mat. Nauk. **10**, 123-127 (1955)
2. Mann, W.R.: Mean value methods in iteration. Am. Math. Soc. **4**, 506-510 (1953)
3. Dong, Q.L., Yuan, H.B.: Accelerated Mann and CQ algorithms for finding a fixed point of a nonexpansive mapping. Fixed Point Theory Appl. **2015**, 125 (2015)
4. Dong, Y.D., Sun, M.D.: New accerleration factors of the Krasnosel'skiĭ-Mann iteration. Results Math. **77**, 194 (2022). https://doi.org/10.1007/s00025-022-01729-x
5. Alvarez, F.: Weak convergence of a relaxed and inertial hybrid projection-proximal point algorithm for maximal monotone operators in Hilbert space. SIAM J. Optim. **14**(3), 773-782 (2004)
6. Matsushita, S.Y.: On the convergence rate of the Krasnosel'skiĭ-Mann iteration. Bull. Aust. Math. Soc. **96**(1), 162-170 (2017)
7. Sahu, D.R.: Applications of accelerated computational methods for quasi-nonexpansive operators to optimization problems. Soft Comput. **24**, 17887-17911 (2020). https://doi.org/10.1007/s00500-020-05038-9
8. Izuchukwu, C., Shehu, Y.: Inertial reflected Krasnosel'skiĭ-Mann iteration with applications to monotone inclusion problems. Optimization. Under review.
9. Eckstein, J., Svaiter, B.F.: A family of projective splitting methods for the sum of two maximal monotone operators. Math. Program. **111**, 173–199 (2008). https://doi.org/10.1007/s10107-006-0070-8
10. Dong, Y.D.: New inertial factors of the Krasnosel'skiĭ-Mann iteration. Set-Valued Var. Anal. **29**(1), 145-161 (2021). https://doi.org/10.1007/s11228-020-00541-5